The Hafnian of Toeplitz Matrices and Perfect Matchings of Arc and Chord Diagrams

(Dmitry Efimov, published July 10 2020)

1 Introduction

Let $A = (a_{ij})$ be a symmetric matrix of even order n over a commutative associative ring. The *hafnian* of A is defined as

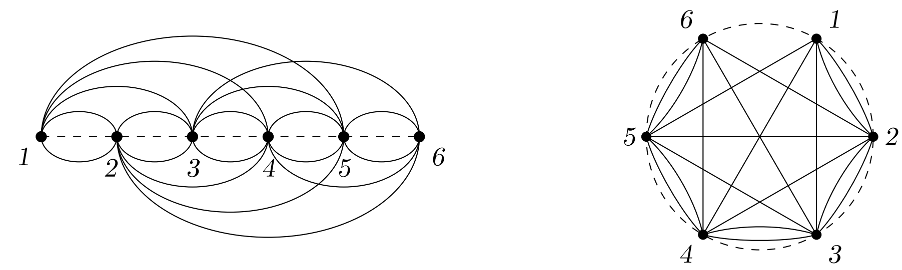
$$\mathrm{Hf}(A) = \sum_{(i_1 i_2|\ldots|i_{n-1} i_n)} a_{i_1 i_2} \cdots a_{i_{n-1} i_n},$$

where the sum runs over all partitions of the set $\{1, 2, \ldots, n\}$ into disjoint pairs $(i_1 i_2)$, $\ldots$, $(i_{n-1} i_n)$ up to the order of pairs, and the order of elements in each pair. So, if $n = 4$ then $\mathrm{Hf}(A) = a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23}$. The hafnian of the empty matrix is taken to be 1.

Recall that a matrix is called *Toeplitz* if all elements of every of its diagonals parallel to the main one are the same. A symmetric Toeplitz matrix is uniquely determined by its first row. Let a, b, c be real or complex numbers. We denote by $T_m(a, b, c)$ a symmetric Toeplitz matrix of order $2m$ with zero main diagonal, whose first row has the form $(0, a, b, b, \ldots, b, c)$ or $(0, a)$ if $m = 1$. For example (dots denote zeros),

$$T_2(a, b, c) = \begin{pmatrix} \cdot & a & b & c \\ a & \cdot & a & b \\ b & a & \cdot & a \\ c & b & a & \cdot \end{pmatrix}.$$

It is easy to see that if M is the adjacency matrix of an unordered multigraph with even number of vertices, then $\mathrm{Hf}(M)$ equals the total number of perfect matchings of the multigraph. We denote by $G_m(a, b, c)$ a multigraph with $2m$ vertices whose adjacency matrix is $T_m(a, b, c)$. It is convenient to represent such a multigraph in the form of an arc or chord diagram. An *arc diagram* is a graph presentation method where all the vertices are located along a line in the plane, while all edges are drawn as arcs. The vertices of a *chord diagram* are located on a circle and edges are chords of the circle. However, if a pair of vertices of a chord diagram is joined by several edges, then to distinguish them in a figure, we will draw them not in the form of segments, but also in the form of arcs. By construction, the vertices 1 and $2m$ of the diagram $G_m(a, b, c)$ are joined by c edges, vertices with numbers differing by one are joined by a edges, and all other pairs of vertices are joined by b edges (see Figure 1).



(a) (b)

Figure 1: Arc (a) and chord (b) diagrams $G_3(2,1,0)$

2 The hafnian of three-parameter Toeplitz matrices

Let $Q_{k,n}$ denote the set of all unordered k-element subsets of the set $\{1,2,\dots,n\}$. Let A be a matrix of order n and $\alpha \in Q_{k,n}$. We denote by $A[\alpha]$ the submatrix of A formed by the rows and columns of A with numbers in α, and by $A\{\alpha\}$ the submatrix of A formed from A by removing the rows and columns with numbers in α. The following property proved in [1]:

Proposition 1. *Let A, B be symmetric matrices of even order n. Then*

$$\mathrm{Hf}(A+B) = \sum_{k=0}^{n/2} \sum_{\alpha \in Q_{2k,n}} \mathrm{Hf}(A[\alpha])\mathrm{Hf}(B\{\alpha\}). \tag{1}$$

Consider the matrix $T_m(a,b,c)$, $m \geq 2$. For brevity, we denote it now by A_m:

$$A_m = \underbrace{\begin{pmatrix} 0 & a & b & \cdots & b & c \\ a & \ddots & \ddots & \ddots & & b \\ b & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & b \\ b & & \ddots & \ddots & \ddots & a \\ c & b & \cdots & b & a & 0 \end{pmatrix}}_{2m}, \quad m \geq 2.$$

This matrix can be represented as the sum of the following two matrices:

$$B_m = \begin{pmatrix} 0 & a & b & \cdots & \cdots & b \\ a & \ddots & \ddots & \ddots & & \vdots \\ b & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & b \\ \vdots & & \ddots & \ddots & \ddots & a \\ b & \cdots & \cdots & b & a & 0 \end{pmatrix}, \quad C_m = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & c-b \\ \vdots & \ddots & & & & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & \ddots & \vdots \\ c-b & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

It has shown in [2], that if we put $0^0 = 1$ then the hafnian of B_m can be calculated by the following formula:

$$\mathrm{Hf}(B_m) = \sum_{k=0}^{m} (a-b)^{m-k} b^k \frac{(m+k)!}{k!(m-k)!2^k}. \tag{2}$$

Another words, the value of the hafnian of such a matrix is equal to the value of the polynomial

$$p_m(x,y) = \sum_{k=0}^{m} \frac{(m+k)!}{k!(m-k)!} \left(\frac{x}{2}\right)^k y^{m-k}$$

in two variables x, y at $x = b$ and $y = a - b$. Note that for $y = 1$ this polynomial coincides with the Bessel polynomial of degree m [3]. Using (1), we find

$$\mathrm{Hf}(A_m) = \mathrm{Hf}(B_m + C_m) = \sum_{k=0}^{m} \sum_{\alpha \in Q_{2k,2m}} \mathrm{Hf}(B_m[\alpha])\mathrm{Hf}(C_m\{\alpha\}).$$

If $\alpha = (1, 2, \ldots, 2m)$, then $C_m\{\alpha\}$ is the empty matrix and $\mathrm{Hf}(C_m\{\alpha\}) = 1$. If $\alpha = (2, 3, \ldots, 2m-1)$, then $\mathrm{Hf}(C_m\{\alpha\}) = \mathrm{Hf}(C_m[1, 2m]) = c - b$. In all other cases, $\mathrm{Hf}(C_m\{\alpha\}) = 0$. It follows that

$$\begin{aligned} \mathrm{Hf}(A_m) &= \mathrm{Hf}(B_m) + (c-b)\mathrm{Hf}(B_{m-1}) \\ &= \sum_{k=0}^{m} (a-b)^{m-k} b^k \frac{(m+k)!}{k!(m-k)!2^k} + (c-b) \sum_{l=0}^{m-1} (a-b)^{m-l-1} b^l \frac{(m+l-1)!}{l!(m-l-1)!2^l}. \end{aligned} \tag{3}$$

Performing simple transformations, we finally obtain

$$\begin{aligned} \mathrm{Hf}(A_m) = {} & \frac{(2m)!}{m!} \left(\frac{b}{2}\right)^m \\ & + \sum_{k=0}^{m-1} \frac{(a-b)^{m-k-1}}{k!} \left(\frac{b}{2}\right)^k \left((a-b)\frac{(m+k)!}{(m-k)!} + (c-b)\frac{(m+k-1)!}{(m-k-1)!} \right). \end{aligned} \tag{4}$$

If $a \neq b$, then the first summand can also be entered under the sum sign:

$$\mathrm{Hf}(A_m) = \sum_{k=0}^{m} \frac{(a-b)^{m-k-1}}{k!} \left(\frac{b}{2}\right)^k \left(\frac{(m+k-1)!}{(m-k)!} \big(m(a+c-2b) + k(a-c)\big) \right). \tag{5}$$

The obtained formulas allow us to give an asymptotic estimate for $\mathrm{Hf}(A_m)$.

Proposition 2. *If* $b = 0$, *then* $\mathrm{Hf}(A_m) = a^{m-1}(a + c)$. *If* $b \neq 0$, *then*

$$\mathrm{Hf}(A_m) \sim \frac{(2m)!}{m!} \left(\frac{b}{2}\right)^m e^{(a-b)/b}, \quad m \to \infty.$$

Proof. If $b = 0$, then nonzero summands in (3) correspond only to the values $k = 0$ and $l = 0$. Therefore, in this case $\mathrm{Hf}(A_m) = a^{m-1}(a + c)$.

In the case $b \neq 0$, the proof is similar, with slight modifications, to the proof of the asymptotic formula for Bessel polynomials in [4]. We introduce for convenience the notation

$$f_m(x, y) = \frac{(2m)!}{m!} \left(\frac{x}{2}\right)^m e^{y/x}.$$

Our task is to show that $\mathrm{Hf}(A_m) \sim f_m(b, a - b)$ as $m \to \infty$. In the course of the proof, we also denote for convenience $\mathrm{Hf}(B_m)$ by S_m. If we replace k by $m - k$ under the summation sign in (2), and then take out the first summand as a common factor, we get:

$$S_m = \sum_{k=0}^{m} (a - b)^k b^{m-k} \frac{(2m - k)!}{k!(m - k)!2^{m-k}} = \frac{b^m(2m)!}{m!2^m} \sum_{k=0}^{m} \frac{(a - b)^k 2^k m!(2m - k)!}{b^k k!(2m)!(m - k)!}.$$

By induction, it can be proved that

$$0 \leq \frac{1}{k!} \left(1 - \frac{2^k m!(2m - k)!}{(2m)!(m - k)!}\right) \leq \frac{1}{2(k - 2)!(2m - 1)}, \quad 2 \leq k \leq m.$$

Hence,

$$\begin{aligned} \left| S_m - \frac{b^m(2m)!}{m!2^m} \sum_{k=0}^{m} \frac{(a - b)^k}{b^k k!} \right| & \\ \leq \frac{|b|^m(2m)!}{m!2^m} \sum_{k=2}^{m} \frac{|a - b|^k}{|b|^k k!} \left(1 - \frac{2^k m!(2m - k)!}{(2m)!(m - k)!}\right) & \\ \leq \frac{|b|^m(2m)!}{m!2^{m+1}(2m - 1)} \sum_{k=2}^{m} \frac{|a - b|^k}{|b|^k (k - 2)!} & \\ \leq \frac{|b|^{m-2}(2m)!|a - b|^2}{m!2^{m+1}(2m - 1)} e^{|a-b|/|b|}. & \end{aligned}$$

Similarly,

$$
\begin{aligned}
S_{m-1} = \sum_{k=0}^{m-1} (a-b)^k b^{m-k-1} \frac{(2m-k-2)!}{k!(m-k-1)!2^{m-k-1}} \\
= \frac{b^m (2m)!}{m! 2^m} \sum_{k=0}^{m-1} \frac{(a-b)^k 2^{k+1} m! (2m-k-2)!}{b^{k+1} k! (2m)! (m-k-1)!} \\
= \frac{b^m (2m)!}{m! 2^m} \sum_{k=0}^{m-1} \left(\frac{(a-b)^k}{b^{k+1} k! (2m-k-1)} \prod_{i=0}^{k} \frac{2m-2i}{2m-i} \right).
\end{aligned}
$$

It follows that

$$
|S_{m-1}| \le \frac{|b|^{m-1}(2m)!}{m! 2^m m} \sum_{k=0}^{m-1} \frac{|a-b|^k}{|b|^k k!} \le \frac{|b|^{m-1}(2m)!}{m! 2^m m} e^{|a-b|/|b|}.
$$

Now we can write the following chain of inequalities:

$$
\begin{aligned}
|\mathrm{Hf}(A_m) - f_m(b, a-b)| = \left| S_m + (c-b) S_{m-1} - \frac{b^m (2m)!}{m! 2^m} \sum_{k=0}^{\infty} \frac{(a-b)^k}{b^k k!} \right| \\
\le \left| S_m - \frac{b^m (2m)!}{m! 2^m} \sum_{k=0}^{m} \frac{(a-b)^k}{b^k k!} \right| + |c-b||S_{m-1}| + \frac{|b|^m (2m)!}{m! 2^m} \sum_{k=m+1}^{\infty} \frac{|a-b|^k}{|b|^k k!} \\
\le f_m(|b|, |a-b|) \left(\frac{|a-b|^2}{2(2m-1)|b|^2} + \frac{|c-b|}{m|b|} + \frac{|a-b|^{m+1}}{(m+1)!|b|^{m+1}} \right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left| \frac{\mathrm{Hf}(A_m)}{f_m(b, a-b)} - 1 \right| = \frac{|\mathrm{Hf}(A_n) - f_m(b, a-b)|}{|f_m(b, a-b)|} \\
\le \frac{f_m(|b|, |a-b|)}{|f_m(b, a-b)|} \left(\frac{|a-b|^2}{2(2m-1)|b|^2} + \frac{|c-b|}{m|b|} + \frac{|a-b|^{m+1}}{(m+1)!|b|^{m+1}} \right).
\end{aligned} \tag{6}
$$

The ratio $f_m(|b|, |a-b|)/|f_m(b, a-b)|$ is a constant, and the expression in brackets approaches zero as $m \to \infty$. This completes the proof. □

3 Examples

3.1 Toeplitz matrices $T_m(2, 1, 1)$

Let a_m denote the value of the hafnian of $T_m(2, 1, 1)$. For example,

$$
a_1 = \mathrm{Hf}(T_1(2, 1, 1)) = \mathrm{Hf} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 2,
$$

$$a_2 = \mathrm{Hf}(T_2(2,1,1)) = \mathrm{Hf}\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} = 2\cdot 2 + 1\cdot 1 + 1\cdot 2 = 7.$$

From (5) we obtain

$$a_m = \mathrm{Hf}(T_m(2,1,1)) = \sum_{k=0}^{m} \frac{1}{k!2^k}\frac{(m+k)!}{(m-k)!}. \tag{7}$$

Setting $a_0 = 1$ and applying (7) for consecutive m, we get the sequence $A001515$ from [5] (the first column in Table 1).

Consider the arc diagram $G_m(2,1,1)$. Neighboring vertices are joined in it by two arcs (we call them conditionally "upper" and "lower" arc), and any other pair of vertices is joined by one arc (see Figure 2). In view of the Introduction, the number of perfect matchings of $G_m(2,1,1)$ equals a_m. Thus, we have a new interpretation of the sequence $A001515$.

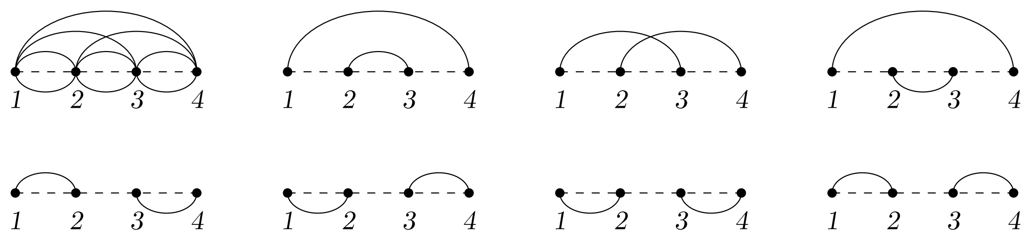


Figure 2: The arc diagram $G_2(2,1,1)$ and its perfect matchings

3.2 Toeplitz matrices $T_m(2,1,2)$

Let b_m denote the value of the hafnian of $T_m(2,1,2)$. For example,

$$b_1 = \mathrm{Hf}(T_1(2,1,2)) = \mathrm{Hf}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 2,$$

$$b_2 = \mathrm{Hf}(T_2(2,1,2)) = \mathrm{Hf}\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix} = 2\cdot 2 + 1\cdot 1 + 2\cdot 2 = 9.$$

From (5) we obtain

$$b_m = \mathrm{Hf}(T_m(2,1,2)) = m\sum_{k=0}^{m} \frac{1}{k!2^{k-1}}\frac{(m+k-1)!}{(m-k)!}, \quad m \geq 2. \tag{8}$$

By Proposition 2

$$b_m \sim \frac{(2m)!}{m!2^m}e.$$

Setting $b_0 = 1$ and using (8) for consecutive m, we get the sequence presented in the second column of Table 1.

Consider the chord diagram $G_m(2,1,2)$. Neighboring vertices are joined in it by two chords, and any other pair of vertices is joined by one chord (see Figure 3). In view of the Introduction, the number of perfect matchings of $G_m(2,1,2)$ equals b_m. It is easy to see, that sequences (a_m) and (b_m) are connected to each other by the following relation:

$$b_m = a_m + a_{m-1}. \tag{9}$$

Indeed, the diagram $G_m(2,1,2)$ differs from $G_m(2,1,1)$ by only one edge joining the vertices 1 and $2m$. For $G_m(2,1,2)$, the number of perfect matchings, in which vertices 1 and $2m$ are joined by an edge, equals a_{m-1}. Hence, b_m is greater than a_m by a_{m-1}.

It is known that the sequence (a_m) satisfies the following recurrence relation:

$$a_m = (2m-1)a_{m-1} + a_{m-2}, \quad a_1 = 2,\ a_0 = 1. \tag{10}$$

From the equalities (9) and (10), we can derive the following recurrence relation for b_m:

$$b_{m+1} = 2mb_m + (2m-2)b_{m-1} + b_{m-2}, \quad m \geq 4.$$

and

$$b_{m+1} = \frac{(4m^2-3)b_m + (2m+1)b_{m-1}}{2m-1}, \quad m \geq 3.$$

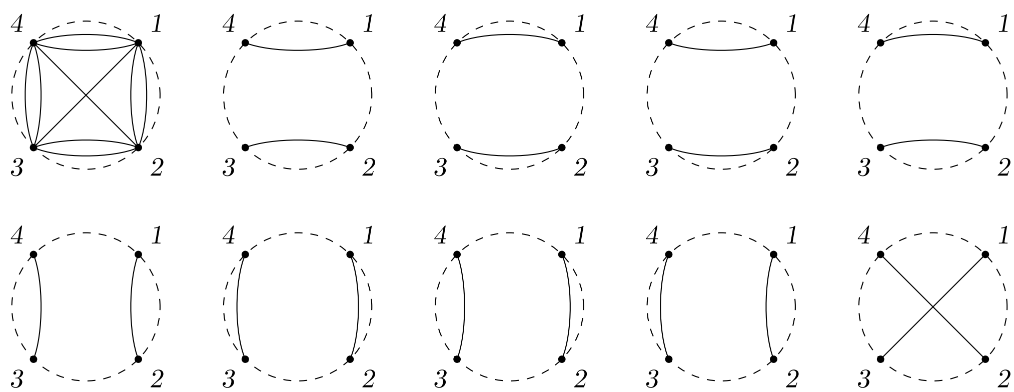


Figure 3: The chord diagram $G_2(2,1,2)$ and its perfect matchings

3.3 Toeplitz matrices $T_m(2,1,0)$

Let c_m denote the value of the hafnian of $T_m(2,1,0)$. For example,

$$c_1 = \mathrm{Hf}(T_1(2,1,0)) = \mathrm{Hf}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 2,$$

$$c_2 = \mathrm{Hf}(T_2(2,1,0)) = \mathrm{Hf}\begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} = 2\cdot 2 + 1\cdot 1 + 0\cdot 2 = 5.$$

From (5) we obtain

$$c_m = \mathrm{Hf}(T_{2m}(2,1,0)) = \sum_{k=1}^{m} \frac{1}{(k-1)!2^{k-1}} \frac{(m+k-1)!}{(m-k)!}, \quad m \geq 2. \tag{11}$$

By Proposition 2

$$c_m \sim \frac{(2m)!}{m!2^m} e.$$

Setting $c_0 = 1$ and using (11) for consecutive m, we get the sequence presented in the third column of Table 1.

Consider the arc diagram $G_m(2,1,0)$. Neighboring vertices are joined in it by two arcs, the vertices 1 and $2m$ are not adjacent if $m \geq 2$, and all other pairs of vertices are joined by one arc (see Figure 4). In view of the Introduction, the number of perfect matchings of $G_m(2,1,0)$ equals c_m. It is easy to see that sequences (a_m) and (c_m) are connected to each other by the following relation:

$$c_m = a_m - a_{m-1}. \tag{12}$$

Indeed, the diagram $G_m(2,1,1)$ differs from $G_m(2,1,0)$ by only one edge joining the vertices 1 and $2m$. For $G_m(2,1,1)$, the number of perfect matchings, in which vertices 1 and $2m$ are joined by an edge, equals a_{m-1}. Hence, a_m is greater than c_m by a_{m-1}. From the equalities (10) and (12), one can derive the following recurrence relation for terms c_m:

$$c_{m+1} = (2m+2)c_m - (2m-4)c_{m-1} - c_{m-2}, \quad m \geq 4.$$

and

$$c_{m+1} = \frac{(4m^2+1)c_m + (2m+1)c_{m-1}}{2m-1}, \quad m \geq 3.$$

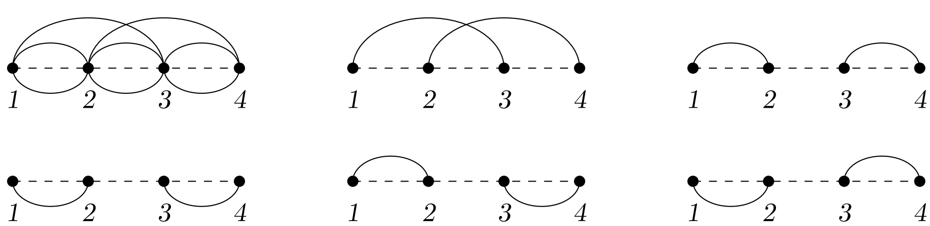


Figure 4: The arc diagram $G_2(2,1,0)$ and its perfect matchings

Starting from the second term, the sequence (c_m) coincides with the sequence $A144498$ from [5]. Thus, one can say that we get a new interpretation of the sequence $A144498$.

3.4 Toeplitz matrices $T_m(1,2,2)$

Let u_m denote the value of the hafnian of $T_m(1,2,2)$. For example,

$$u_1 = \mathrm{Hf}(T_1(1,2,2)) = \mathrm{Hf}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1,$$

$$u_2 = \mathrm{Hf}(T_2(1,2,2)) = \mathrm{Hf}\begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix} = 1\cdot 1 + 2\cdot 2 + 2\cdot 1 = 7.$$

From (5) we derive that

$$u_m = \mathrm{Hf}(T_{2m}(1,2,2)) = \sum_{k=0}^{m} \frac{(-1)^{m-k}}{k!}\frac{(m+k)!}{(m-k)!}. \tag{13}$$

By Proposition 2

$$u_m \sim \frac{(2m)!}{\sqrt{e}\, m!}. \tag{14}$$

Setting $u_0 = 1$ and using (13) for consecutive m, we get the sequence presented in the first column of Table 2. Elements of this sequence coincide in absolute value with the corresponding elements of the sequence $A002119$ from [5].

Consider the arc diagram $G_m(1,2,2)$. Neighboring vertices are joined in it by one arc, and any other pair of vertices is joined by two arcs (see Figure 5). In view of the Introduction, the number of perfect matchings of $G_m(1,2,2)$ equals u_m.

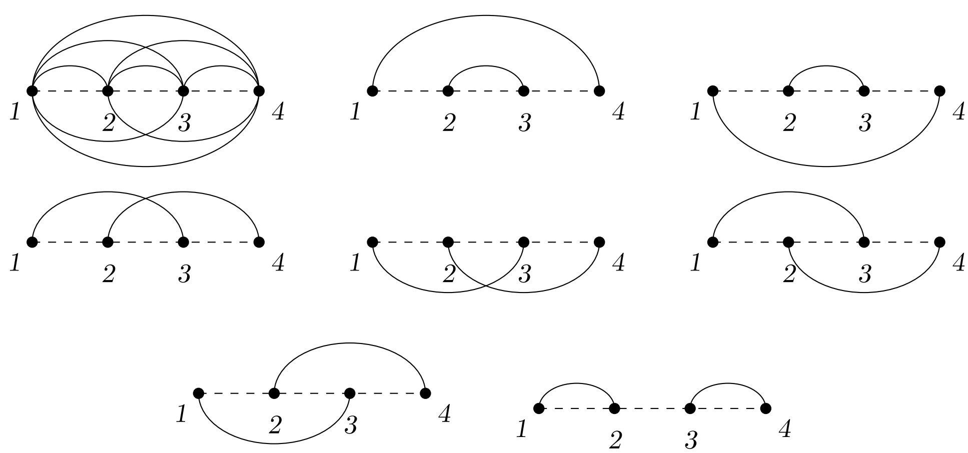


Figure 5: The arc diagram $G_2(1,2,2)$ and its perfect matchings

Now we derive a recurrence for the sequence (u_m). For $G_m(1,2,2)$, consider perfect matchings, in which the vertex $2m$ is joined by an arc with the vertex $2m-1$. It is obvious

that the number of such perfect matchings equals u_{m-1}. Consider a perfect matching, in which the vertex $2m$ is joined by an "upper" arc with the vertex $2m-2$ (see Figure 6(a)). The remaining $2m-2$ vertices can be paired in at least u_{m-1} ways. But the vertices $2m-1$ and $2m-3$ are considered here as neighboring and therefore assume only one variant of the connection (by an "upper" arc), although if we consider the diagram $G_{2m}(1,2,2)$ in general, they can be joined by two different arcs. Thus, one must also take into account perfect matchings, in which the vertices $2m-1$ and $2m-3$ are joined by a "lower" arc (see Figure 6(b)). The number of such matchings is obviously u_{m-2}.

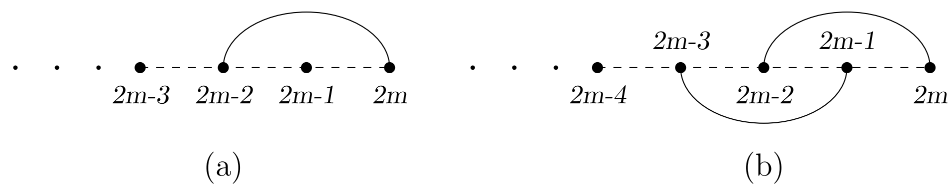


Figure 6: A derivation of a recurrence for (u_m)

The same is true for perfect matchings, in which the vertex $2m$ is joined with the vertex $2m-2$ by a "lower" arc. Thus, the number of perfect matchings, in which vertices $2m$ and $2m-2$ are joined by an arc, equals $2(u_{m-1}+u_{m-2})$. Continuing to reason in a similar way and summing over all possible variants of arcs incident to the vertex $2m$, we obtain

$$u_m + u_{m-1} = (4m-2)u_{m-1} + (4m-6)u_{m-2} + \cdots + 10u_2 + 6.$$

On the other hand, applying the given formula to u_{m-1}, we arrive at the equality:

$$u_{m-1} + u_{m-2} = (4m-6)u_{m-2} + (4m-10)u_{m-3} + \cdots + 10u_2 + 6.$$

Substituting this expression into the previous one, we finally obtain

$$u_m = (4m-2)u_{m-1} + u_{m-2}, \quad u_1 = 1,\ u_0 = 1. \tag{15}$$

3.5 Toeplitz matrices $T_m(1,2,1)$

Let v_m denote the value of the hafnian of $T_m(1,2,1)$. For example,

$$v_1 = \mathrm{Hf}(T_1(1,2,1)) = \mathrm{Hf}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1,$$

$$u_2 = \mathrm{Hf}(T_2(1,2,1)) = \mathrm{Hf}\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} = 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 = 6.$$

From (5) we get

$$v_m = \mathrm{Hf}(T_{2m}(1,2,1)) = 2m \sum_{k=0}^{m} \frac{(-1)^{m-k}}{k!} \frac{(m+k-1)!}{(m-k)!}, \quad m \geq 2. \tag{16}$$

By Proposition 2

$$v_m \sim \frac{(2m)!}{\sqrt{e}\, m!}.$$

Taking $v_0 = 1$ and using (16) for consecutive $m \geq 2$, we get the sequence presented in the second column of Table 2.

Consider the chord diagram $G_m(1,2,1)$. Neighboring vertices are joined in it by one chord, and any other pair of vertices is joined by two chords (see Figure 7). In view of the Introduction, the number of perfect matchings of $G_m(1,2,1)$ equals v_m.

It is easy to see that sequences (u_m) and (v_m) are connected to each other by the following relation:

$$v_m = u_m - u_{m-1}. \tag{17}$$

Indeed, the diagram $G_m(1,2,2)$ differs from $G_m(1,2,1)$ only by one edge joining the vertices 1 and $2m$. Hence, u_m is greater than v_m by the number of perfect matchings of $G_m(1,2,2)$, in which the vertices 1 and $2m$ are joined by an edge, i.e., by u_{m-1}. From equalities (15) and (17), one can derive the following recurrence relations for terms v_m:

$$v_{m+1} = (4m+3)v_m - (4m-7)v_{m-1} - v_{m-2}, \quad m \geq 4.$$

and

$$v_{m+1} = \frac{8m^2 v_m + (2m+1)v_{m-1}}{2m-1}, \quad m \geq 3.$$

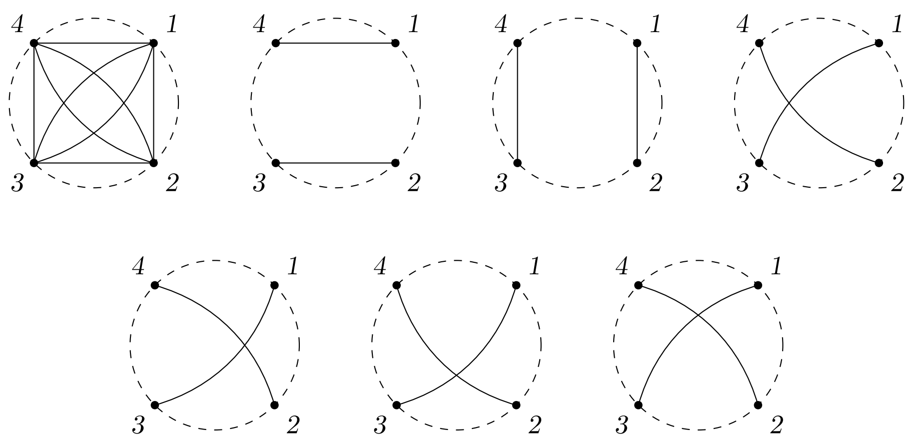


Figure 7: The chord diagram $G_2(1,2,1)$ and its perfect matchings

3.6 Toeplitz matrices $T_m(1,2,0)$

Let w_m denote the value of the hafnian of $T_m(1,2,0)$. For example,

$$w_1 = \mathrm{Hf}(T_1(1,2,0)) = \mathrm{Hf}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1,$$

$$w_2 = \mathrm{Hf}(T_2(1,2,0)) = \mathrm{Hf}\begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} = 1 \cdot 1 + 2 \cdot 2 + 0 \cdot 1 = 5.$$

From (5) we get

$$w_m = \mathrm{Hf}(T_{2m}(1,2,0)) = \sum_{k=0}^{m} \frac{(-1)^{m-k-1}}{k!} \left[\frac{(m+k-1)!}{(m-k)!}(-3m+k) \right], \quad m \geq 2. \tag{18}$$

Putting $w_0 = 1$ and using (18) for consecutive, we get the sequence represented in the third column of Table 2. By Proposition 2

$$w_m \sim \frac{(2m)!}{\sqrt{e}\, m!}.$$

Consider the arc diagram $G_m(1,2,0)$. Neighboring vertices are joined in it by one arc, the vertices 1 and $2m$ are not adjacent if $m \geq 2$, and all other pairs of vertices are joined by two arcs (see Figure 8). In view of the Introduction, the number of perfect matchings of $G_m(1,2,0)$ equals w_m.

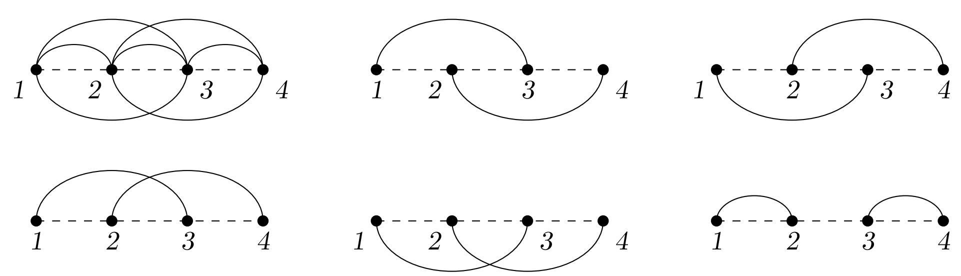


Figure 8: The arc diagram $G_2(1,2,0)$ and its perfect matchings

It is not hard to see that sequences (u_m) and (w_m) are linked to each other by the following relationship:

$$w_m = u_m - 2u_{m-1}, \quad m \geq 2. \tag{19}$$

Indeed, the diagram $G_m(1,2,2)$ differs from $G_m(1,2,0)$ by two arcs joining the vertices 1 and $2m$. Hence, u_m is greater than w_m by twice the number of perfect matchings of $G_m(1,2,2)$,

in which the vertices 1 and $2m$ are joined by an arc, i.e., by $2u_{m-1}$. From equalities (15) and (19), we can derive the following recurrence relations for w_m:

$$w_{m+1} = (4m+4)w_m - (8m-13)w_{m-1} - 2w_{m-2}, \quad m \geq 4.$$

and

$$w_{m+1} = \frac{(32m^2 - 12m + 2)w_m + (8m+1)w_{m-1}}{8m-7}, \quad m \geq 3.$$

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This article was published in the OEIS, July 10 2020

m	$\mathrm{Hf}(T_m(2,1,1))$	$\mathrm{Hf}(T_m(2,1,2))$	$\mathrm{Hf}(T_m(2,1,0))$
0	1	1	1
1	2	2	2
2	7	9	5
3	37	44	30
4	266	303	229
5	2431	2697	2165
6	27007	29438	24576
7	353522	380529	326515
8	5329837	5683359	4976315
9	90960751	96290588	85630914
10	1733584106	1824544857	1642623355
11	36496226977	38229811083	34762642871
12	841146804577	877643031554	804650577600
13	21065166341402	21906313145979	20224019536825
14	569600638022431	590665804363833	548535471681029
15	16539483668991901	17109084307014332	15969883030969470
16	513293594376771362	529833078045763263	496754110707779461

Table 1: The value of the hafnian of matrices $T_m(a,b,c)$

m	$\mathrm{Hf}(T_m(1,2,2))$	$\mathrm{Hf}(T_m(1,2,1))$	$\mathrm{Hf}(T_m(1,2,0))$
0	1	1	1
1	1	1	1
2	7	6	5
3	71	64	57
4	1001	930	859
5	18089	17088	16087
6	398959	380870	362781
7	10391023	9992064	9593105
8	312129649	301738626	291347603
9	10622799089	10310669440	9998539791
10	403978495031	393355695942	382732896853
11	16977719590391	16573741095360	16169762600329
12	781379079653017	764401360062626	747423640472235
13	39085931702241241	38304552622588224	37523173542935207
14	2111421691000680031	2072335759298438790	2033249827596197549
15	122501544009741683039	120390122318741003008	118278700627740322977
16	7597207150294985028449	7474705606285243345410	7352204062275501662371

Table 2: The value of the hafnian of matrices $T_m(a,b,c)$